Application of Conduction Transfer Functions and Periodic Response Factors in Cooling Load Calculation Procedures

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ABSTRACT

This paper presents an overview of the conduction transfer function (CTF) and periodic response factor (PRF) methods of calculating conductive heat transfer. Different forms of the equations used in cooling load calculations are compared and contrasted. Particular attention is given to the methods included in the ASHRAE Loads Toolkit. The toolkit contains the source code for ASHRAE's new load calculation methods, the heat balance method (HBM) and the radiant time series method (RTSM). Each method uses a similar, but different, conduction calculation technique. The HBM uses CTFs and the RTSM uses PRFs. Since there are limited numbers of CTFs and PRFs in the literature, the toolkit algorithms provide a means of calculating CTFs and PRFs for stand-alone computer programs or for generating CTF and PRF libraries. This paper describes the CTF and PRF algorithms in the toolkit and demonstrates implementation of the toolkit modules in a program that calculates CTFs and PRFs.

INTRODUCTION

In order to effectively use the ASHRAE cooling load procedures, it is necessary to understand and correctly apply conduction transfer functions (CTFs) or periodic response factors (PRFs) to the conductive heat transfer calculation. Although response factor and transfer function methods are well established in the literature (Stephenson and Mitalas 1971; Hittle 1979; Ceylan and Myers 1980; Seem 1987; Ouyang and Haghighat 1991), misconceptions persist concerning their application to cooling load procedures. Several forms of the equations relating to different boundary conditions are shown in the literature. Methods of calculating the coefficients differ, and their accuracy is not easily checked.

The objective of this paper is to reconcile the various forms of the transfer function equations, discuss implicit assumptions associated with each form, and illustrate by way of an example calculation the use of the various methods. Particular attention is given to the conduction transfer function methods presented in the ASHRAE Loads Toolkit. An algorithm that uses the toolkit CTF module is presented along with a simple program to generate CTFs and PRFs for use in cooling load procedures.

The heat balance method (HBM) is the standard ASHRAE load calculation method as described in the ASHRAE Handbook—Fundamentals (2001). This method is based on simultaneously satisfying a system of equations that includes a zone air heat balance and a set of outside and inside heat balances at each surface/air interface. The system of equations may be solved in a computer program using successive substitution, Newton techniques, or (with linearized radiation) matrix methods.

The radiant time series method (Spitler et al. 1997) is a simplified method that does not solve the heat balance equations. The method is “heat-balance based” to the extent that the storage and release of energy in the zone is approximated by a predetermined zone response, called “radiant time factors” (RTFs). By incorporating these simplifications, the RTSM calculation procedure becomes explicit, avoiding the requirement to solve the simultaneous system of heat balance equations. The method is useful not only for peak load calculations but also for estimating component contributions to the hourly cooling loads. If the radiant time factors and the periodic response factors for a particular zone configuration are known, the RTSM may be implemented in a spreadsheet.

In both the HBM and the RTSM, two simplifying assumptions are made in solving the wall heat conduction problem.
First, heat conduction is assumed to be one-dimensional. Two-
dimensional effects due to corners and nonuniform boundary
conditions are neglected. Second, materials are assumed to be
homogeneous and have constant thermal properties. As a
result, the diffusion equation of conductive heat transfer
problem is simplified as shown in Equation 1. With the use of
Fourier’s law (Equation 2) for calculating conductive heat
flux, Equations 1 and 2 are the governing equations of conduc-
tive heat transfer problems in cooling load calculation.

\[
\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} = 0
\]  
\[
q'' = -k \frac{\partial^2 T(x,t)}{\partial x^2}
\]  

Although the one-dimensional, transient conduction
problem can be solved analytically, the analytical solution is
immediately complicated when the analysis is extended to
multi-layered constructions. Analytical solutions for multi-
layered slabs require special mathematical functions and
complex algebra. Ultimately, numerical methods must be
employed at some level to solve the problem. Solution tech-
niques include lumped parameter methods, frequency
response methods, finite difference or finite element methods,
and Z-transform methods (McQuiston et al. 2000). The toolkit
implements Laplace and state-space methods for calculating
conduction transfer functions (CTFs) and provides an algo-
rithm to derive periodic response factors (PRFs) from a set of
conduction transfer functions.

CTFs and PRFs are dependent only on material properties
and reflect the transient response of a given construction for
any set of environmental boundary conditions. Since material
properties are typically assumed to be constant in HVAC ther-
mal load calculations, it is possible to pre-calculate these coef-
ficients. Although CTF and PRF coefficients for typical
constructions are available in the ASHRAE Handbook—
Fundamentals (2001) and Spitler and Fisher (1999b), the
ASHRAE Loads Toolkit (Pedersen 2001), makes it possible to
quickly and accurately construct a stand-alone computer
program that will calculate CTFs and PRFs for any arbitrary
wall configuration. This paper presents an algorithm for pre-
calculating these coefficients using the toolkit modules.

FORMULATIONS OF
TRANSFER FUNCTION EQUATIONS

The transfer function equations for conduction calcula-
tion are formulated differently in load calculation methods.
The HBM uses conduction transfer functions (CTFs), while
the RTSM uses periodic response factors (PRFs). In the HBM,
the instantaneous conduction flux is represented by a simple
linear equation that relates the current rate of conductive heat
transfer to temperature and flux histories, while in RTSM, the
conduction flux is a linear function of temperatures only.

Conduction Transfer Function (CTF) Formulations

The CTF formulation of the surface heat fluxes involves
four sets of coefficients. Following Spitler’s nomenclature
(McQuiston et al. 2000) X, Z, and Y are used to represent the
exterior, interior, and cross terms, respectively. Equation 3a
shows the zeroth outside and cross terms operating on the
current hour’s surface temperatures. \(H_{out}\) is the flux history
term as shown in Equation 3b. Together the current hour’s
surface temperatures and the history term yield the total flux
at the outside surface.

\[
q_{ko,0} = -Y_{0}T_{ox,0} + X_{0}T_{ox,0} + H_{out}
\]  

where

\[
H_{out} = - \sum_{n=1}^{N_x} Y_n T_{ox,0 - n\delta} + \sum_{n=1}^{N_z} X_n T_{ox,0 + n\delta} + \sum_{n=1}^{N_y} \delta \phi_{ko,0 - n\delta}
\]  

Likewise, Equations 4a and 4b show the flux at the inside
surface.

\[
q_{ki,0} = -Z_0 T_{oi,0} + Y_0 T_{oi,0} + H_{in}
\]  

where

\[
H_{in} = - \sum_{n=1}^{N_x} Z_n T_{oi,0 - n\delta} + \sum_{n=1}^{N_z} Y_n T_{oi,0 + n\delta} + \sum_{n=1}^{N_y} \delta \phi_{ki,0 - n\delta}
\]  

As indicated in Equations 3 and 4, the current heat fluxes
are closely related to the flux histories. The flux histories,
shown as constant terms in Equations 3 and 4, are not only
related to previous surface temperatures but also related to
previous heat fluxes. Equations 3a and 3b or Equations 4a and
4b are usually solved iteratively with an assumption that all
previous heat fluxes are equal at the beginning of the iteration.
The converged solution produces flux history terms \(H_{out}\) and
\(H_{in}\) that correctly account for the thermal capacitance of a
given construction.

The temperatures operated on by the conduction transfer
functions may be either surface or air temperatures. “Surface-
to-surface” CTFs, which operate on surface temperatures and
are required by the heat balance method, have the advantage of
allowing for variable convective heat transfer coefficients.
“Air-to-air” CTFs operate between either the sol-air tempera-
ture or the air temperature on the outside and the air setpoint
temperature on the inside. Air-to-air CTFs include the appro-
priate film coefficients as resistive layers in the wall assembly.
As shown in Figure 1, surface-to-surface CTFs are represented
by the thermal circuit between \(T_{ox}\) and \(T_{oi}\) while air-to-air
CTFs are represented by the thermal circuit between \(T_{i}\) and \(T_{o}\).
For constructions with the same material layer arrangement
and properties, the surface-to-surface CTFs are always the
same, while air-to-air CTFs differ depending on the selected
values of the film coefficients.

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The 1997 ASHRAE Handbook—Fundamentals presents an “air-to-air” conduction equation that includes additional simplifications. The $b$ and $c$ terms shown in Equation 5 operate on the sol-air temperature and the constant room air temperature, respectively.

$$q'' = \sum_{n=0}^{6} b_n T_{e,0} - n \delta - \sum_{n=1}^{6} d_n q''_{e,0} - n \delta - T_{rc} \sum_{n=0}^{6} c_n$$  \hspace{1cm} (5)

It should be noted that Equation 5 is suitable only for load calculations. Historically, it was used in the Transfer Function Method (TFM) (McQuiston and Spitler 1992) and can be used without loss of generality in the Radiant Time Series Method (RTSM).

Although Equations 3 through 5 are solutions to the transient, one-dimensional conduction problem, it is useful to consider the steady-state limit of these equations. Under steady-state conditions, the exterior and interior heat fluxes are equal and the following identities are readily apparent (Equation 6):

$$\sum_{n=0}^{N_s} X_n = \sum_{n=0}^{N_s} Y_n = \sum_{n=0}^{N_i} Z_n \quad or \quad \sum_{n=0}^{6} b_n = \sum_{n=0}^{6} c_n$$  \hspace{1cm} (6)

In combination with the standard formulation for steady-state heat transfer through a wall ($q'' = U \Delta T$), an expression for $U$, the overall heat transfer coefficient, in terms of conduction transfer functions can be derived as shown in Equation 7.

$$U = \frac{\sum_{n=0}^{N_s} Y_n}{N_s} \quad or \quad U_f = \frac{\sum_{n=0}^{6} b_n}{6}$$  \hspace{1cm} (7)

**Periodic Response Factor (PRF) Formulations**

As formulated in the ASHRAE Loads Toolkit, the Radiant Time Series Method for design load calculations uses periodic response factors (PRFs) (also called “conduction time factors”) rather than CTFs to calculate conductive heat transfer through walls and roofs. PRFs operate only on temperatures; the current surface heat flux is a function only of temperatures and does not rely on previous heat fluxes, as shown in Equation 8.

$$q'' = \sum_{j=0}^{23} P_j (T_{e,0} - j \delta - T_{rc})$$  \hspace{1cm} (8)

This formulation is premised on the steady, periodic nature of the sol-air temperature over a 24-hour period (Spitler et al. 1997). Although the number of PRFs may vary, the 24 PRFs shown in Equation 8 correspond to 24 hourly changes in the sol-air temperature for a single diurnal cycle. It is clear from Equation 8 that the overall heat transfer coefficient, $U$, is represented by the sum of the periodic response factors as shown in Equation 9.

$$U = \sum_{j=0}^{23} P_j$$  \hspace{1cm} (9)

The periodic response factor directly scales the contribution of previous fluxes (in the form of temperature gradients) to the current conductive heat flux. As a result, the periodic response factor series provides a visual representation of the thermal response of the wall. As shown in Figure 2, wall 17 has a slower thermal response than roof 10 because it is a more thermally massive construction.

PRFs are directly related to CTFs as shown in Equation 10 (Spitler and Fisher 1999a) and may be derived directly from CTFs. The toolkit uses this method to calculate periodic response factors.

$$P = d^4 b$$  \hspace{1cm} (10)

where
As shown in Equation 10, the PRFs are related to the cross and flux CTF terms. The first column of the \( P \) matrix is the resulting PRFs, \( P_0, P_1, P_2, \ldots, P_{23} \). Since the sol-air temperature is used in RTSM conduction calculations, the \( b \) and \( d \) matrices must be filled with air-to-air CTFs. This eliminates the surface heat balance calculations in HBM. However, if conductive heat transfer is an isolated concern, the PRFs can be calculated from surface-to-surface CTFs. This reflects the actual conduction response of a construction without considering the outside and inside film coefficients.

**IMPLICIT ASSUMPTIONS OF TRANSFER FUNCTION EQUATIONS**

The assumptions behind the transfer function equations come from the CTF calculation methods. Two widely used CTF calculation methods are the Laplace method (Stephenson and Mitalas 1971; Hittle 1979) and the state-space method (Ceylan and Myers 1980; Seem 1987; Ouyang and Haghighat 1991). A brief overview of these two methods is included in the following sections.

**Laplace Transform Method**

Hittle (1979) introduced a procedure to solve the conductive heat transfer governing Equations 1 and 2 by using the Laplace transform method. The system in the Laplace domain is shown in Equation 14.

\[
\begin{bmatrix}
P_0 & \cdots & P_5 & P_4 & P_3 & P_2 & P_1 \\
P_1 & \cdots & P_5 & P_4 & P_3 & P_2 & P_1 \\
P_2 & \cdots & P_5 & P_4 & P_3 & P_2 & P_1 \\
P_3 & \cdots & P_5 & P_4 & P_3 & P_2 & P_1 \\
P_4 & \cdots & P_5 & P_4 & P_3 & P_2 & P_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{23} & \cdots & P_4 & P_3 & P_2 & P_1 & P_0
\end{bmatrix}
\]

Response factors are generated by applying a unit triangular temperature pulse to the inside and outside surface of the multi-layered slab. The response factors are defined as an infinite series of discretized heat fluxes on each surface due to both an outside and inside temperature pulse. Hittle also described an algebraic operation to group response factors into CTFs, and to truncate the infinite series of response factors by the introduction of flux history coefficients. A convergence criterion shown in Equation 15 is used in the Laplace method to determine whether the numbers of CTFs and flux history coefficients are sufficient such that the resulting CTFs accurately represent the response factors.

\[
\begin{bmatrix}
q_{t_k}(s) \\
q_{ko}(s)
\end{bmatrix} = \begin{bmatrix}
\frac{D(s)}{B(s) B(s)} - \frac{1}{B(s) B(s)} T_i(s) \\
\frac{1}{B(s) B(s)} D(s) T_o(s)
\end{bmatrix} \tag{14}
\]

As shown in Equation 10, the PRFs are related to the cross and flux CTF terms. The first column of the \( P \) matrix is the resulting PRFs, \( P_0, P_1, P_2, \ldots, P_{23} \). Since the sol-air temperature is used in RTSM conduction calculations, the \( b \) and \( d \) matrices must be filled with air-to-air CTFs. This eliminates the surface heat balance calculations in HBM. However, if conductive heat transfer is an isolated concern, the PRFs can
and there is no limited number of CTF terms in this approach. Peavy (1978) suggests that the number of flux CTF terms should always be less than or equal to 5, even for thermally massive walls.

**State-Space Method**

The use of the state-space method in solving the governing Equations 1 and 2 was introduced by Seem (1987). The state-space expression is formulated by using either finite-difference or finite-element methods to discretize the governing equations. The state-space expression relates the interior and exterior boundary temperatures to the inside and outside surface heat fluxes at each node of a multi-layered slab as shown in Equations 17 and 18.

\[
\begin{align*}
\frac{dT_{is}}{dt} & = a_T \begin{bmatrix} T_{is} \\ T_{os} \end{bmatrix} + b_T \begin{bmatrix} T_i \\ T_o \end{bmatrix} \\
\frac{dT_{os}}{dt} & = \begin{bmatrix} T_{is} \\ T_{os} \end{bmatrix} + c_T \begin{bmatrix} T_i \\ T_o \end{bmatrix}
\end{align*}
\]

(17)

The exterior and interior temperature variations, \(T_i\) and \(T_o\), are modeled with piecewise linear functions. Equations 17 and 18 can be simplified using some matrix algebraic calculations, so that the surface heat fluxes are directly related to the surface and boundary temperatures only. This system of equations is solved directly for CTFs, without calculating response factors.

The number of CTF terms is increased until the ratio of the last to the first flux history coefficient is less than a tolerance limit.

**COMPARISON OF RESULTS FROM LAPLACE AND STATE-SPACE METHODS**

Since Laplace and state-space CTFs are calculated differently, the resulting CTFs are expected to be different even for the same slab. The number of CTF terms, and/or the numerical value of each single CTF can be different. Table 1 lists the CTFs from the Laplace and the state-space methods for ASHRAE roof number 10 (ASHRAE 2001). The Roof 10 construction is an eight-layer construction consisting of membrane, sheathing, insulation board, metal deck, and suspended acoustical ceiling. Note that as presented in the Handbook, the inside and outside layers of these construction types are resistance layers that model the combined effects of radiation and convection on inside and outside surfaces, respectively; the resulting CTFs are therefore air-to-air type. The data on the right-hand side of the tables are calculated from the Laplace method, while the data on the left-hand side are from the state-space method. The summation of each CTF series and the U-factors are also shown in the tables for comparison. Conduction transfer functions are not unique for any construction. The differences can be in terms of the number of CTF terms and/or numerical values. However, under steady-state condition, these resulting CTFs will predict the same U-factor. As shown in Table 1, both CTFs from Laplace and state-space methods predict nearly the same overall heat transfer coefficient.

The number of CTF terms generated by the toolkit algorithms is determined by the thermal mass of the construction materials. Table 2 compares Roof 10 CTFs to the

<table>
<thead>
<tr>
<th>CTFs</th>
<th>ASHRAE Toolkit (State-space)</th>
<th>ASHRAE Toolkit (Laplace)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.84341408E-02 1.55869365E+00</td>
<td>1.79231787E-02 1.55760048E+00</td>
</tr>
<tr>
<td>1</td>
<td>1.52709767E-01 -1.53385043E+00</td>
<td>1.52697088E-01 -1.52985220E+00</td>
</tr>
<tr>
<td>2</td>
<td>6.91431835E-02 2.24087760E-01</td>
<td>7.00275192E-02 2.21313276E-01</td>
</tr>
<tr>
<td>3</td>
<td>2.16248562E-03 -6.48395624E-03</td>
<td>2.25362542E-03 -6.15402138E-03</td>
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<tr>
<td>4</td>
<td>1.98067596E-06 4.43484168E-06</td>
<td>-- --</td>
</tr>
<tr>
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<td>2.34795161E-15 --</td>
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<td>2.42901412E-01 2.42907532E-01</td>
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<tr>
<td>U-factors</td>
<td>3.79868921E-01</td>
<td>3.79862500E-01</td>
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</table>

<table>
<thead>
<tr>
<th>Number of CTF terms</th>
<th>Roof 10</th>
<th>Wall 17</th>
<th>Roof 10</th>
<th>Wall 17</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3.79868921E-01</td>
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<td>3.79862500E-01</td>
<td>7.22032100E-01</td>
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</tbody>
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**Table 1. Calculated Conduction Transfer Functions for Roof 10 by Different Methods**

**Table 2. Comparison of CTFs for Different Construction and Calculation Methods**
Wall 17 (ASHRAE 2001) CTFs. Wall 17 is a six-layer construction consisting of brick, insulation board, and brick. Note that the more thermally massive wall 17 has eight CTF terms, while the relatively lightweight Roof 10 has six. The more thermally massive the construction, the greater the number of CTF terms regardless of the CTF solution technique used.

Although the maximum number of terms reported in the 1997 ASHRAE Handbook—Fundamentals is seven, thermally massive constructions may require more terms for accurate calculations (Giaconia and Orioli 2000). The toolkit algorithms address this problem by automatically selecting an appropriate number of CTF terms. Note also from Table 2 that the number of CTF terms is also related to the the method of calculation. The toolkit Laplace method generates less CTF terms than the state-space method. However, the resulting overall heat transfer coefficients are nearly the same.

The PRFs of Roof 10 can be derived according to Equations 10 to 13. Using the Laplace CTFs listed in Table 1, the resulting \( \mathbf{d} \) and \( \mathbf{b} \) matrices are shown below.

\[
\begin{bmatrix}
1 & 0 & \ldots & \ldots & \ldots & -2.95287743E-02 & 3.90083134E-01 \\
3.90083134E-01 & 1 & 0 & \ldots & \ldots & 0 & -2.95287743E-02 \\
-2.95287743E-02 & 3.90083134E-01 & 1 & 0 & \ldots & \ldots & 0 \\
0 & -2.95287743E-02 & 3.90083134E-01 & 1 & 0 & \ldots & 0 \\
0 & 0 & -2.95287743E-02 & 3.90083134E-01 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & -2.95287743E-02 & 3.90083134E-01 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.79231787E-02 & 0 & \ldots & \ldots & \ldots & 2.25362542E-03 & 7.00275192E-02 & 1.52697088E-01 \\
1.52697088E-01 & 1.79231787E-02 & 0 & \ldots & \ldots & 2.25362542E-03 & 7.00275192E-02 & 1.52697088E-01 \\
7.00275192E-02 & 1.52697088E-01 & 1.79231787E-02 & 0 & \ldots & \ldots & 2.25362542E-03 & 7.00275192E-02 \\
2.25362542E-03 & 7.00275192E-02 & 1.52697088E-01 & 1.79231787E-02 & 0 & \ldots & \ldots & 0 \\
0 & 2.25362542E-03 & 7.00275192E-02 & 1.52697088E-01 & 1.79231787E-02 & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 2.25362542E-03 & 7.00275192E-02 & 1.52697088E-01 & 1.79231787E-02 & \ldots
\end{bmatrix}
\]
After taking the inverse of \( \mathbf{d} \) and calculating \( \mathbf{d}^{-1} \mathbf{b} \), the \( \mathbf{P} \) matrix is calculated. Table 3 lists the first column of the \( \mathbf{P} \) matrix, i.e., PRFs of Roof 10 from Laplace CTFs. The overall heat transfer coefficients were calculated using Equation 9 and are also shown for comparison. The values only differ at the eighth significant digit compared to the overall heat transfer coefficients calculated using Equation 7 and shown in Tables 1 and 2. While the CTFs can be different for the same slab, PRFs represent the construction thermal response and theoretically must be unique for any slab. Table 3 illustrates that the PRFs for Roof 10 are the same when calculated from Laplace and state-space CTFs.

### Table 3. Periodic Response Factors of Roof 10 (W\( \cdot \)m\(^2\)\( \cdot \)K\(^{-1}\))

<table>
<thead>
<tr>
<th></th>
<th>ASHRAE Toolkit (State-space)</th>
<th>ASHRAE Toolkit (Laplace)</th>
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<td>( P_0 )</td>
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<td>( P_{14} )</td>
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</table>

U-factors 3.79868925E-01 3.79862470E-01

The numeric differences for the corresponding PRFs shown in Table 3 are to be expected since both the methods and the convergence criteria used to determine the state-space and Laplace CTFs are different. The accuracy of the node number criterion in using calculating state-space CTFs does not necessarily match the convergence criterion of the root finding procedure in the Laplace CTF calculation. In addition, some differences due to computer round-off error when converting CTFs to PRFs are expected.

### USING THE TOOLKIT MODULES TO CALCULATE CTFS AND PRFS

Each ASHRAE Loads Toolkit module contains a set of subroutines and functions required to achieve its computational objective. The following sections describe the incorporation of the CTF modules in a stand-alone program that generates CTFs and PRFs.

#### Overview of the Algorithm

The algorithm consists of a driver program that first calls the toolkit CTF routines to calculate CTFs and then converts the CTFs to PRFs by applying Equation 10. The required surface construction information is in standard toolkit input file format and is read using the toolkit input processor as described by Crawley et al. (1998). The overall procedure is shown in Figure 3 and can be summarized as follows:

1. Get the surface and construction information from the input file.
2. Pass this information to the CTF computational module and calculate CTFs.
3. Print the CTF results to an output file.
4. Convert the CTFs to PRFs.
5. Print the resulting PRFs to another output file.
The structure of the algorithm for CTF and PRF calculations is shown in Figure 4. The “USE” statement is a Fortran 90 keyword that makes the subroutines in a FORTRAN 90 module available to a program, subroutine, or another module. “USE” is followed by the toolkit module name. Subroutines in one FORTRAN 90 module cannot be “called” by another module unless the calling routine or module “uses” the target module. The module “InputProcessor” handles all input data. The data are organized under keywords with a one-to-one correspondence between each definition and data value. The keywords for calculating CTFs and PRFs are SURFACE, CONSTRUCTION, and MATERIAL_LAYER. Table 4 shows the required data for each keyword. By changing the data values of these keywords, the toolkit generates different CTFs and PRFs. Although manual construction of a toolkit input data file is tedious, the structured input format is conducive to the application of a graphical user interface as discussed in the next section.

The function “getNumObjectsFound” and the subroutine “GetObjectItem” in this module are used in the CTF and PRF calculations. “getNumObjectsFound” returns the number of surfaces in the calculations, while “GetObjectItem” returns two arrays containing all data values (numeric and alpha) for the keyword “Surface.” Although there will be a number of surface data values returned from this subroutine, only the surface name and the construction ID are useful for the calculations. The argument variables of this subroutine are described in the toolkit documentation (Pedersen 2001). In a similar way, the data associated with the keywords CONSTRUCTION and MATERIAL_LAYER are read into the program.

The program can be configured to use either the Laplace or the state-space CTF module. Figure 4 shows the program configured to use the “StateSpaceCTFCalc” module. It contains subroutines that import the construction and material data and calculate the CTFs and PRFs.

### Table 4. Required Data Values in the Toolkit Input File for CTF and PRF Calculation

<table>
<thead>
<tr>
<th>Keyword</th>
<th>Required Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>SURFACE</td>
<td>Surface name; construction ID</td>
</tr>
<tr>
<td>CONSTRUCTION</td>
<td>Construction name; layer names</td>
</tr>
<tr>
<td>MATERIAL_LAYER</td>
<td>Layer names; thickness, thermal conductivity, density, specific heat or thermal resistance</td>
</tr>
</tbody>
</table>

**Figure 4** Structure of the toolkit CTF and PRF calculations.

![Diagram](image-url)
layer data, generates CTFs, and performs related calculations. The subroutine “CalcStateSpaceConduction” is the main calling routine of this module. It was modified to obtain the desired output format of the CTFs and to facilitate calculation of PRFs. The inputs to this subroutine are surface number and construction names. The surface number is assigned in order of appearance of the surface data in the input file; construction names are identifiers for each set of data and are used to map the construction data to a specific surface in the computational algorithm.

The inputs to the PRF subroutine are the number of CTF terms and the cross and flux CTFs. These data are used to fill the $b$ and $d$ matrices shown in Equation 10. FORTRAN matrix functions are used to calculate the PRFs and the results are printed to an output file. If the PRFs are to be used in the RTSM, air resistance layers must be included at both inside and outside of the construction specification in the input file. Once the air resistance layers have been added, the resulting CTFs and PRFs will be air-to-air type, as described earlier. Typical inside and outside resistance values for vertical surfaces are 0.12 and 0.04 (m$^2$·K·W$^{-1}$) or 0.68 and 0.25 (h·ft$^2$·°F·Btu$^{-1}$), respectively (ASHRAE 2001). The air-to-air PRFs can be directly applied to the RTSM; however, the air-to-air CTFs cannot be used in the HBM since the inside and outside convection coefficients are already included in the heat balance calculations.

THE TOOLKIT USER INTERFACE TO CTF AND PRF GENERATOR

To facilitate calculation of CTFs and PRFs and illustrate the utility of the ASHRAE Loads Toolkit, a simple user interface to the FORTRAN 90 program was developed. This interface and a compiled version of the program can be downloaded from www.hvac.okstate.edu and freely used for any purpose. The toolkit program was compiled into a dynamic link library (DLL) file. When this file is called from the interface, it imports data from the input file, generates CTFs and PRFs, and prints them to separate output files. The inputs to this DLL file are the names of the toolkit idd and idf files. The interface allows the user to display the output on a spreadsheet or notepad for further application or analysis. In addition, related information, including surface names, constructions, and U-factors, is displayed by the interface.

The dialog box shown in Figure 5 guides the user in creating a valid toolkit input file by requesting definition of surface names, number of layers, and material properties. The number of material layers is limited by the toolkit algorithm to ten. Material layer data are entered from outside to inside. Material layers can be either resistive, as illustrated by layer F01, or completely specified, as illustrated by layer G03 in Figure 5. It should be noted that only fully specified layers capture the expected “lag and decrement” effect of thermal mass. The resistance layer option should only be used for material layers with low thermal capacitance. This primarily applies to air layers but is often also applied to glazing.

Figure 5  Dialog box used for creating toolkit input file (IP).
The default layer type of fully specified properties, can be changed by clicking the “Edit” button. For input convenience, an ASHRAE material database, which contains the data shown in Table 22, chapter 29, 2001 ASHRAE Handbook—Fundamentals, is included as shown in Figure 6. This database can be modified and saved for future reference. The interface can handle up to 100 surfaces, which may be specified in either SI or IP units. All IP unit data are converted to SI units by the interface before writing the input file, since input to the toolkit modules must be in consistent SI units and in accordance with toolkit conventions.

The PRF generator can efficiently and conveniently calculate the desired CTFs and PRFs for any construction. This section establishes the validity of the program in terms of the self-consistency of the program outputs, the agreement of program output with previously published data, and the conservation of energy based on a steady-state test.

Consistency of PRFs and CTFs. Since the U-factor is unique for any construction and since it can be calculated either using CTFs or PRFs, the U-factor check can be used to evaluate the program algorithm that converts CTFs to PRFs. Figure 7 compares the U-factors calculated by CTFs and PRFs.
for the wall and roof database used by Spitler and Fisher (1999b). The diagonal line represents a zero percent difference between the two. The results show nearly perfect agreement between the CTF and PRF U-factors. The differences show up in the fifth decimal place and are probably due to round-off error.

Comparison with Published Data. The second step in evaluating program output is based on the fact that the PRF series is unique for any construction regardless of whether it is derived from Laplace or state-space CTFs. Since a PRF database for the walls and roofs in the ASHRAE Handbook—Fundamentals is already available in the literature (Spitler and Fisher 1999b), the validity of the program outputs is evaluated by a term-by-term comparison of calculated and published PRFs for each construction, as shown in Figure 8. The small differences shown in the figure could be caused by slight numerical differences in the input data, the type of CTFs used to derive PRFs, as well as the round-off error in both calculations. In addition, if the convergence criterion used in the CTF calculation is different from that used by Spitler and Fisher (1999b), it would also cause the numerical difference in the PRFs. In general, the program outputs agree very well with the published data.

Steady-State Evaluation. The program outputs for a given construction can also be verified at the steady-state limit by comparing the U-factor predicted by the CTFs or PRFs with that predicted by the steady-state calculation. At steady state, the U-factor is calculated as shown in Equation 19.

\[ U = \frac{1}{R_T} \]  

(19)

Therefore, the total thermal resistance is calculated using Equation 22.

\[ R_T = \sum_{i=1}^{n} R_i \]  

(22)

If the calculated U-factor from the steady-state Equation 19 is equal to that from the CTFs and PRFs, energy is conserved. Figure 9 shows the comparison of the program output U-factors with the steady-state U-factors based on the same construction database used by Spitler and Fisher (1999b). The U-factors shown on the vertical axis are the PRF U-factors from Figure 7.

Figure 9 shows that use of the program outputs to calculate U-factors is satisfactory. The steady-state U-factors agree with the calculated U-factors to within ±3.4%. Although some round-off error is expected, the root cause of the differences is primarily due to the convergence criteria used in the Toolkit module. The CTFs calculated from the computer program are based on the default Toolkit settings. A better result is obtained if one tightens the convergence criteria in the Toolkit CTF module.

The steady-state evaluation is necessary but not sufficient to guarantee the accuracy of the transient calculation. A more rigorous transient evaluation would apply a sinusoidal temperature variation to the outside surface and a constant temperature to the inside surface and compare the resulting heat flux to the analytical solution presented in ASHRAE RP-1052 (Spitler et al. 2001). However, for most standard constructions, the steady-state test is a good indicator of CTF and PRF accuracy.

SUMMARY AND CONCLUSIONS

Transfer function equations continue to provide a robust, accurate, and tractable approach to calculating conductive heat gains in cooling load procedures. This paper illustrates two commonly used formations in conduction calculation, i.e., CTF and PRF formulations. The application and the rela-
tion between CTF and PRF are discussed. Although care must be taken to consistently apply CTFs and PRFs, depending on whether they were generated with or without a convective resistance layer, the application of boundary conditions and solution techniques is straightforward and consistent.

Implicit assumptions of the two CTF calculation methods (Laplace and state-space) imply that CTFs can be different for the same material construction. The differences are in terms of number of CTF terms and the CTF numeric values. However, the predicted overall heat transfer coefficients are the same. PRFs represent the thermal response of a material construction and therefore are unique regardless of the calculation methods. An example comparing the Laplace and state-space methods showed that the more thermally massive construction (Wall 17) carries more CTF terms and has slower thermal response than Roof 10.

In the past, the most serious drawback to the use of transfer function and response factor methods was the complexity of the computer code required to generate the coefficients. TheASHRAE Loads Toolkit addresses this problem by providing the source code required to generate conduction transfer functions and periodic response factors for arbitrary wall or roof constructions. The computational algorithm required to implement the toolkit modules in a CTF/PRF generator program was presented in this paper. In addition, input/output and interface issues were discussed. The CTFs and PRFs calculated by the Toolkit algorithms can be directly applied to heat balance and radiant time series load calculation procedures.

The outputs of the computer program were evaluated based on the physical significance of the CTFs and PRFs and were compared to the published literature. A simple method for checking the steady-state accuracy of CTFs and PRFs was also used in the evaluation. The results showed that based on the steady-state check, program outputs are within ±3.4% of the calculated U-factor.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>a, b, c, d</td>
<td>coefficient matrices that depend on material properties and/or film coefficients</td>
</tr>
<tr>
<td>b</td>
<td>24×24 air-to-air cross CTF coefficient matrix (Equation 13)</td>
</tr>
<tr>
<td>d</td>
<td>24×24 air-to-air flux CTF coefficient matrix (Equation 12)</td>
</tr>
<tr>
<td>C</td>
<td>material capacitance, J·°C⁻¹ (Btu·°F⁻¹)</td>
</tr>
<tr>
<td>b_n</td>
<td>air-to-air cross CTF coefficient, W·m⁻²·K⁻¹ (Btu·h⁻¹·ft⁻²·°F⁻¹)</td>
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<tr>
<td>c_n</td>
<td>air-to-air interior CTF coefficient, W·m⁻²·K⁻¹ (Btu·h⁻¹·ft⁻²·°F⁻¹)</td>
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<tr>
<td>d_n</td>
<td>air-to-air flux coefficient, dimensionless</td>
</tr>
<tr>
<td>A(s), B(s), D(s)</td>
<td>overall transmission matrices that depend on material properties and/or film coefficients</td>
</tr>
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<td>h_i</td>
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</tr>
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</tr>
<tr>
<td>n</td>
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<tr>
<td>N_x</td>
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<tr>
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<td>number of cross CTF terms</td>
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<tr>
<td>N_z</td>
<td>number of interior CTF terms</td>
</tr>
<tr>
<td>N_b</td>
<td>number of flux CTF terms</td>
</tr>
<tr>
<td>P</td>
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</tr>
<tr>
<td>P_j</td>
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<td>q''</td>
<td>heat flux, W·m⁻² (Btu·h⁻¹·ft⁻²)</td>
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<td>q''_ki</td>
<td>heat flux at interior surface, W·m⁻² (Btu·h⁻¹·ft⁻²)</td>
</tr>
<tr>
<td>q''_ko</td>
<td>heat flux at exterior surface, W·m⁻² (Btu·h⁻¹·ft⁻²)</td>
</tr>
</tbody>
</table>

**Figure 9** Comparison of the PRF generator U-factors to the steady-state U-factors, W·m⁻²·K⁻¹.
\( q''_0 \) = heat flux for the current hour, W·m\(^{-2}\) (Btu·h\(^{-1}\)·ft\(^{-2}\))

\( q''_{c0} \) = heat flux at interior surface, W·m\(^{-2}\) (Btu·h\(^{-1}\)·ft\(^{-2}\))

\( q_{ki}''(s) \) = inside flux terms in the Laplace domain

\( q_{ko}''(s) \) = outside flux terms in the Laplace domain

\( R \) = material resistance, m·K·W\(^{-1}\) (h·ft\(^{2}\)·°F·Btu\(^{-1}\)·in\(^{-1}\))

\( T \) = temperature, °C (°F)

\( T_{e0} \) = sol-air temperature, °C (°F)

\( T_i \) = inside air temperature, °C (°F)

\( T_o \) = sol-air or outside air temperature, °C (°F)

\( T_s \) = inside surface temperature, °C (°F)

\( T_{os} \) = constant room temperature, °C (°F)

\( T_i(s) \) = interior boundary temperature in the Laplace domain.

\( T_o(s) \) = exterior boundary temperature in the Laplace domain.

\( U \) = overall heat transfer coefficient, W·m\(^{-2}\)·K\(^{-1}\) (Btu·h\(^{-1}\)·ft\(^{-2}\)·°F\(^{-1}\))

\( U_f \) = overall heat transfer coefficient with film coefficients, W·m\(^{-2}\)·K\(^{-1}\) (Btu·h\(^{-1}\)·ft\(^{-2}\)·°F\(^{-1}\))

\( x \) = heat flow direction, m (ft)

\( X_n \) = surface-to-surface exterior CTF coefficient, W·m\(^{-2}\)·K\(^{-1}\) (Btu·h\(^{-1}\)·ft\(^{-2}\)·°F\(^{-1}\))

\( Y_n \) = surface-to-surface cross CTF coefficient, W·m\(^{-2}\)·K\(^{-1}\) (Btu·h\(^{-1}\)·ft\(^{-2}\)·°F\(^{-1}\))

\( Z_n \) = surface-to-surface interior CTF coefficient, W·m\(^{-2}\)·K\(^{-1}\) (Btu·h\(^{-1}\)·ft\(^{-2}\)·°F\(^{-1}\))

\( \alpha \) = thermal diffusivity, m\(^2\)·s\(^{-1}\) (ft\(^2\)·s\(^{-1}\))

\( \delta \) = time step

\( \theta \) = time

\( \tau \) = time, s

\( \phi_n \) = flux coefficient, dimensionless

REFERENCES


Pedersen, C.O. 2001. Building loads calculation toolkit. Urbana, IL: Building Systems Laboratory, Department of Mechanical & Industrial Engineering, University of Illinois at Urbana-Champaign.


